N-soliton pattern in a self-gravitating fluid disk

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A nonlinear process of density perturbations in an infinitesimally thin fluid disk with uniform rotation and self-gravitation is investigated. In a locally Cartesian system the amplitude equation turns out to be the nonlinear Schrödinger equation. By solving the equation it is shown that the perturbed density can undergo modulation in space, forming an *N*-soliton pattern in the radial direction. An astrophysical application of this model to the early evolution of Saturn's ring system is studied. [S1063-651X(97)11012-1]

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I. INTRODUCTION

When the density perturbations excited in a self-gravitating fluid disk have finite amplitudes, a modulation of the envelope occurs owing to nonlinear couplings of the perturbations. Then, in general, a spatial pattern is formed in the disk to saturate to the finite amplitude. Pattern formations in nonequilibrium fluid systems have been extensively discussed [1]. When nonlinearities are not strong, it is said that a system can often be described theoretically by simple equations, which are called amplitude equations. The ways of deriving amplitude equations from microscopic equations include the introduction of multiple scales to formally separate the fast and slow dependences [2,3] or the use of mode projection techniques [4,5], which emphasizes the slaving idea. Microscopic equations are considered as either more or less realistic descriptions of the phenomena or mathematical models chosen so that their linear instabilities and long-time solutions mimic those of the system under study. An interesting point is that there are different microscopic models that have the same type of amplitude equations, whose form is universal and whose numerical parameters reflect the details of each physical system. The studies of pattern formations are encountered in solid-state physics, nonlinear optics, chemistry, and biology, showing the well-known analogies to fluid systems. This collective process of pattern formation is also important in some astrophysical backgrounds [6], especially in the case of a self-gravitating disk. To deal with such a problem some simplifications must be made for it to be analytically tractable. In our model the disk is assumed to be infinitely thin and rotating uniformly. This idealized model has been studied in our previous work and the amplitude equation has been obtained by two-time-scale analysis [7]. In this paper we will investigate the amplitude equation in a circular symmetric case, which turns out to be the famous nonlinear Schrödinger equation.

The nonlinear Schrödinger equation has been solved by Zakharov and Shabat [8] using the inverse scattering method. It is shown that for the initial value problem with the natural boundary condition, the long-time asymptotic solution is *N* stable solitons. This indicates that the nonlinear interactions of the finite amplitude perturbations lead to the pattern formation from an initially smooth envelope, forming an *N*-soliton structure in the disk.

The application of our model to the highly complicated system of Saturn's rings gives some interesting results. The study of Saturn's rings is usually based on the resonance theory of celestial mechanics. However, for plasma physics or gravitating systems with long-distance interactions, the role of collective effects is often decisive. The analysis of collective effects is a tradition of plasma physics. Introduction of its ideas and methods to the study of the gravitating systems seems fruitful [6]. Accordingly, we will try to explain the origin of Saturn's ring structure on the basis of the amplitude equation describing the pattern formation in a fluid disk.

The remainder of this paper is organized as follows. In Sec. II the amplitude equation describing the interactions of density perturbations up to nonlinear fourth order in an infinitely thin fluid disk with uniform rotation and self-gravitation is obtained. In Sec. III the long-time asymptotic solution to the nonlinear Schrödinger equation is given. The locations of the *N* solitons are calculated under a certain initial condition. The application to Saturn's rings is studied in Sec. IV. Conclusions are given in Sec. V.

II. AMPLITUDE EQUATION

An infinitesimally thin fluid disk with uniform rotation and self-gravitation is described by equations

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \mathbf{v}) = 0, \qquad (1)$$

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$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+2\boldsymbol{\Omega}\times\mathbf{v}=\frac{1}{\sigma}\boldsymbol{\nabla}p-\boldsymbol{\nabla}\Phi+\Omega^2\mathbf{r}, \tag{2}$$

$$\nabla^2\Phi+\frac{\partial^2\Phi}{\partial z^2}=4\pi G\sigma\delta(z), \tag{3}$$

where σ is the surface density, $\mathbf{v}$ is the local velocity, $\boldsymbol{\Omega}$ is the rotating velocity, and Φ is the gravitational potential. The equations are written in the reference frame rotating at the angular velocity $\boldsymbol{\Omega}$.

As our discussion is on the density perturbation of the disk, we write

$$\sigma=\sigma_0+\tau, \tag{4}$$

where σ_0 is the background unperturbed density and τ the perturbation. In the assumption of slow variation of σ_0 with respect to the perturbation τ and using the WKB approximation $k\gg(\partial/\partial r)\ln|\tau|$ in which k is the wave number of the perturbation,

$$\tau\propto\tilde{\tau}\exp(-i\omega t+i\mathbf{k}\cdot\mathbf{r}). \tag{5}$$

After the linearization of the original fluid equations one can obtain the dispersion relation

$$\omega_k^2=4\Omega^2+k^2c_s^2-2\pi G\sigma_0 k, \tag{6}$$

where c_s is the sound velocity.

By writing

$$\mathbf{v}=\frac{\lambda}{\sigma}\boldsymbol{\nabla}\mu+\boldsymbol{\nabla}\varphi-\boldsymbol{\Omega}\times\mathbf{r}, \tag{7}$$

where λ and μ are the Clebsch variables suitable for barotropic fluids [9], Eqs. (1)–(3) can be brought into a Hamiltonian description approach, which is widely discussed in plasma physics [10]. Transforming to the Fourier components and introducing the canonical variable $a_{\mathbf{k}}$, where

$$\tau_{\mathbf{k}}=\frac{k\sigma_0^{1/2}}{(2\omega_{\mathbf{k}})^{1/2}}(a_{\mathbf{k}}+a_{\mathbf{k}}^*), \tag{8}$$

we get the perturbative expansion equation of $a_{\mathbf{k}}$ [7]:

$$\begin{aligned}\frac{\partial a_{\mathbf{k}}}{\partial t}+i\omega_{\mathbf{k}}a_{\mathbf{k}}=&-i\int d\mathbf{k}_1 d\mathbf{k}_2[V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}a_{\mathbf{k}_1}a_{\mathbf{k}_2}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)+2V_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1}a_{\mathbf{k}_1}^*a_{\mathbf{k}_2}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)+V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^*a_{\mathbf{k}_1}^*a_{\mathbf{k}_2}^*\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)]\\&-i\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_2[4W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}a_{\mathbf{k}_1}a_{\mathbf{k}_2}a_{\mathbf{k}_3}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)+6(W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}+W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^*)a_{\mathbf{k}_1}^*a_{\mathbf{k}_2}a_{\mathbf{k}_3}\\&\times\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)+12W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}a_{\mathbf{k}_1}a_{\mathbf{k}_2}^*a_{\mathbf{k}_3}^*\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)+4W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^*a_{\mathbf{k}_1}^*a_{\mathbf{k}_2}^*a_{\mathbf{k}_3}^*\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)]\\&+\cdots.\end{aligned} \tag{9}$$

The most unstable mode lies at $\partial\omega_k/\partial k=v_g=0$ or $k_0=\pi G\sigma_0/c_s^2$, where the dispersion curve has a minimum; for a nearly marginally stable disk with $Q=2c_s\Omega/\pi G\sigma_0\geqslant 1$, this mode is most easily excited. We suppose that the perturbation is a narrow wave packet with central wave vector $\mathbf{k}_0$ (chosen as the direction of the x axis) selected. Thus, in consideration of the linear dispersion (6), only the four-wave process is important. We have written this out in Eq. (9) up to fourth order.

Using the two-time-scale method, from Eq. (9), we can derive the amplitude equation (for details see [7])

$$i\left(\frac{\partial A}{\partial t}+v_g\frac{\partial A}{\partial x}\right)+\frac{1}{2}v_g'\frac{\partial^2 A}{\partial x^2}+\frac{1}{2}\frac{v_g}{k_0}\nabla_\perp^2 A-(2\pi)^2 T|A|^2 A=0, \tag{10}$$

where

$$\tau=\left(\frac{2\sigma_0}{\omega_{k_0}}\right)^{1/2}k_0\mathrm{Re}[A(\mathbf{r},t)e^{ik_0x-i\omega_{k_0}t}], \tag{11}$$

$$v_g=\left(\frac{\partial\omega}{\partial k}\right)_{k_0},\quad v_g'=\left(\frac{\partial^2\omega}{\partial k^2}\right)_{k_0}, \tag{12}$$

and T is the interaction coefficient. In a locally Cartesian system, making the transformations [11]

$$\frac{\partial}{\partial x}\to\frac{\partial}{\partial r},\quad \nabla_\perp\equiv\frac{\partial}{\partial y}\to\frac{1}{r}\frac{\partial}{\partial\theta} \tag{13}$$

and taking $v_g\approx 0$ for the most unstable mode, we get the equation

$$i\frac{\partial A}{\partial t}+\frac{1}{2}v_g'\frac{\partial^2 A}{\partial r^2}-(2\pi)^2 T|A|^2 A=0, \tag{14}$$

which describes the original perturbations in the circular symmetric case.

Because $T<0$ and $v_g'>0$ [7], Eq. (14) can be transformed into a dimensionless form

$$i\frac{\partial u}{\partial s}+\frac{1}{2}\frac{\partial^2 u}{\partial\hat{x}^2}+|u|^2u=0, \tag{15}$$

where $u=2\pi|T|^{1/2}s_0^{1/2}A$, $\hat{x}=r/(v_g's_0)^{1/2}$, $s=t/s_0$, and s_0 is an appropriate time scale. If we analytically continue Eq. (15) to $\hat{x}<0$ in a symmetrical way, it becomes the well-known nonlinear Schrödinger equation.

III. *N*-SOLITON SOLUTION

Zakharov and Shabat [8] have shown that the nonlinear Schrödinger equation can be solved exactly by the inverse scattering method for the initial value problem with the natural boundary condition that $|u|$ tends to zero as $|\hat{x}|\to\infty$ together with all its $\hat{x}$ derivatives. Satsuma and Yajima [12] provided a detailed analysis. It is found that if the initial value of Eq. (15) is real and not an antisymmetric function of $\hat{x}$, which is just our case, a bound soliton state is formed. The long-time asymptotic behavior of the solution is well described by a series of solitons, while the nonsoliton part is shown to decay as $s^{-1/2}$.

For brevity we will not display how to solve the nonlinear Schrödinger equation, but summarize the related results. The long-time asymptotic solution is written as [8]

$$u\to\sum_{n=1}^{N} S_n(\hat{x},s)=2\eta_n\,\mathrm{sech}[2\eta_n(\hat{x}-2\xi_n s-\hat{x}_n)]\times\exp[-2i\xi_n\hat{x}+2i(\xi_n^2-\eta_n^2)s], \tag{16}$$

where

$$\hat{x}_n=(2\eta_n)^{-1}\ln\left(\frac{b(\zeta_n)}{2\eta_n a'(\zeta_n)}\right). \tag{17}$$

In the above, $\zeta_n=\xi_n+i\eta_n$ (ξ_n and η_n are both real) are discrete eigenvalues of the equation

$$iv_{\hat{x}}+Uv=\zeta\sigma_3 v, \tag{18}$$

where the eigenfunction v is a two-component column vector $v=\binom{v_1}{v_2}$, $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, and $U=\begin{pmatrix}0&u\\u*&0\end{pmatrix}$, in which u is a solution of Eq. (15) and v develops according to

$$iv_t=Av,$$

$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix}\left(\frac{1}{2}\,\frac{\rho^2-1}{\rho^2+1}\,\frac{\partial^2}{\partial\hat{x}^2}-i\zeta\,\frac{\partial}{\partial\hat{x}}+C\right)+(\rho^2+1)^{-1}\begin{pmatrix}\rho^2|u|^2 & iu_{\hat{x}}\\ -i\rho^2u^*_{\hat{x}} & -|u|^2\end{pmatrix}, \tag{19}$$

where ρ and C are constants independent of $\hat{x}$; $a(\zeta)$ and $b(\zeta)$ are the transmission and reflection coefficients for the scattered state of Eq. (18).

It is still rather difficult to deal with Eqs. (18) and (19). An example that can be solved by the inverse-scattering method is the case $u(\hat{x},s=0)=Q\ \mathrm{sech}(\hat{x})$ [12]. In this case, $\xi_n=0$ and $\eta_n=Q-n+\frac{1}{2}$, where n must be positive integers satisfying $Q-n+\frac{1}{2}>0$; the number of solitons is $N=[Q+\frac{1}{2}]$ with the square brackets indicating the integer part. The N-soliton solution becomes

$$u\to\sum_{n=1}^{N} 2\eta_n\mathrm{sech}[2\eta_n(\hat{x}-\hat{x}_n)]\exp[-2i\eta_n^2 s], \tag{20}$$

where

$$\hat{x}_n=(2\eta_n)^{-1}\ln\left(\frac{b(i\eta_n)}{2\eta_n a'(i\eta_n)}\right) \tag{21}$$

and

$$a(\zeta)=\frac{[\Gamma(-i\zeta+\frac{1}{2})]^2}{\Gamma(-i\zeta+Q+\frac{1}{2})\Gamma(-i\zeta-Q+\frac{1}{2})},$$

$$b(\zeta)=\frac{i\sin(\pi Q)}{\cosh(\pi\zeta)}.$$

From above we see that the N solitons are located at $\hat{x}_n$.

IV. A POSSIBLE ASTROPHYSICAL APPLICATION: SATURN'S RING

The radial structure of Saturn's rings has been specified by the high resolutions of spacecraft and ground-based observations. It is found that the rings display the following features: several localized main broad rings (fine structure) with the hyperfine structure (or microstructure) of thousands of narrow rings. Now we try to apply our model to understand Saturn's ring structure. First we will discuss the feasibility of our simple model in this application. Here it is sufficient to mention the extensive review of Nieto [13]. Nieto stated that nebular disks in the solar system mainly undergo two evolution stages: a hydrodynamic stage and a point gravitation stage. According to Nieto, patterns of a regular distribution, for example, the Titius-Bode law, are inherited from the hydrodynamic stage of the gas-dust disk. This means that the general feature of the regular pattern formed in the first stage is not altered seriously in the second stage. We reasonably assume that this is also true for Saturn's system. Then we deal with the proto-Saturn disk in the early hydrodynamic stage when the central object has not been fully developed.

We take an infinitely thin fluid disk with uniform rotation and self-gravitation as an approximate model for the nebula of primeval Saturn. In a sense we attempt to solve the ''easy'' problem: understanding the basic properties of structure formation. The thickness of Saturn's rings is perhaps below 1 km [14]. So when we consider the perturbations with wavelengths much larger than the thickness, the effect of taking into account the finite thickness is not significant in the study of the collective process at larger scales. So the disk can be regarded as an infinitely thin one. In general, the treatment will be incomplete in the sense that it omits the effect of differential rotation. The existence of differential rotation implies a potential source of free energy to tap to feed into growing disturbances [15]. For the problem of the formation of the ring-shaped structure, we concern ourselves with the nonlinear modulation of the finite amplitude perturbations. These perturbations are supposed to have been excited and preserved. In this case the omission of the differential rotation does not seem essential. The role of the assumption of uniform rotation is to make the problem mathematically tractable. In view of the universality of amplitude equations, it is reasonable to think that the overall effect of including differential rotation or other details is encapsulated in the parameters v_g' and T in Eq. (14). So what we really assumed is the constancy of the v_g' and T. In the case of differential rotation, these parameters will depend on distance r in a complicated way. As long as we do not concern the excitation and amplification of the perturbations, it seems plausible to assume that v_g' and T vary slowly in the main part of the disk. Therefore, the main feature of the pattern formation scenario is not altered.

The relatively strong density perturbations are from the central bulge, excited by the accretion process of the formation of the central object (Saturn). When these perturbations propagate outward, their amplitudes gradually decrease owing to collisions or other damping mechanism, showing the validity of the natural boundary condition. Such perturbations are considered to be the seeds of the hyperfine structure. When they enter the nonlinear stage, due to the modulational instabilities, the initial smooth envelope will finally evolve to *N* stable envelope solitons. The solitons have sharp edges, leading to the formation of the structure with modulation scales.

Thus the disk ultimately separates into several primitive broad rings. In the later period the outer rings accrete into satellites, while the inner rings survive due to tidal force from the central planet and develop into the observed pattern. Therefore, the formation of Saturn's rings can be successfully described by a linear scale and a modulation scale of the collective process of the density perturbations in a primordial nebular disk.

Now we give some detailed calculations. The model parameters for Saturn's unperturbed nebular disk are taken as $c_s \sim 2\ \mathrm{cm\ s}^{-1}$, $\sigma_0 \sim 20\ \mathrm{g\ cm}^{-2}$, and $\Omega \sim 1.2\times 10^{-6}\ \mathrm{s}^{-1}$. Then $Q \sim 1.1$, which signifies the marginal global stability of the model disk.

The hyperfine structure is thought to be formed by the seeds of linear perturbations. The characteristic scale is $\lambda \sim 2\pi/k_0 = 2c_s^2/G\sigma_0 \sim 50$ km in our chosen parameters. The typical width of narrow ringlets is said to be a few tens of kilometers [16], showing that the linear scale we obtained agrees with the observation in magnitude.

As we have said, the formation of the fine structure of the rings is supposed to be attributed to the nonlinear process described by the amplitude equation (14). Its solution is *N* solitons. The soliton-shaped profiles characterize the early broad rings. The locations of the *N* solitons rely on the initial form of the envelope, which we do not know. To give an intuitive insight we just assume that the initial envelope can be modeled by $Q\ \mathrm{sech}(r/L)$, where L is taken to be about the magnitude of Saturn's scale (R_s). Then the locations of the solitons can be calculated according to formula (21). If we take $N=9$, the locations of the nine solitons have the property $r_{n+1}/r_n = \hat{x}_{n+1}/\hat{x}_n \sim 1.15-1.35$. These ratios are not dependent on the absolute values of the parameters v_g' and T in Eq. (14). It is found that the above ratios approximate to the Titius-Bode law of Saturn's main rings (A, B, and C rings) and inner six regular satellites: $r_{n+1}/r_n \sim 1.17-1.28$.

V. CONCLUSION

We have investigated the pattern formation process in an infinitely thin fluid disk with self-gravitation and uniform rotation. In the circularly symmetric case, the amplitude equation for the envelope function turns out to be the nonlinear Schrödinger equation. If the natural boundary condition is satisfied by the finite-amplitude perturbation, the long-time asymptotic behavior of the solution to the nonlinear Schrödinger equation is *N* stable solitons. Thus the self-modulation of the initial perturbation leads to the formation of an *N*-soliton structure in the disk. The application of this model to Saturn's ring system is discussed and it is interesting to find that the ring structure can be understood in this soliton pattern picture. However, whether this collective process competes with the resonance or other mechanism remains to be seen.

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